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Analysis of Hybrid Coupled System with Nonlinear Conditions and Real World Applications

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Abstract

We study a coupled hybrid system of fractional differential equations (FDEs) with the help of conformable fractional differential operator (CFDO). We deduce various results related to existence theory of solution, Ulam stability and numerical analysis for the concerned problem. The existence theory is studied by using fixed point theory, while stability analysis is based on the concepts of Ulam- Hyers. In addition, the fractional Runge-Kutta technique of order four (RK-4) is used to compute approximate solution of the mentioned problem. The finding results are demonstrated by a real world application.

Keywords: system of differential problems; theory of fixed point; numerical tools; existence theory; stability theory.

1 Introduction

Currently, fractional calculus (FC) is the hot area of research because of its applications and ability to model real world problems. For instance, Kumar and Baleanu [24] studied applications of FC in physics. Youssef and Ali [52] have studied memory effects produced by fractional time derivative and integral in diffusion like space with wavelet techniques. The mentioned subject basically extend the notion of classical integrals and derivatives to any real or complex number. The idea was first pointed out by Leibnitz and L'Hospital in the seventeenth century but did not attract the interest of the researchers like ordinary derivative due to its complicated nature [11]. However, later when some important function came into the existence, the area has been studied comprehensively by many mathematicians of the world. Later, the subject was given keen interest by various popular mathematicians like, Lacroix, Riemann, Liouville, Lagrange, Euler, Abel, Oldham, Caputo, Spanier, Podlubny, and many more [35]. In the last two decades, this subject has been used for the solution of various mathematical and physical problems, where the classical concepts failed to explain the behavior of some problems in applied sciences. For example, the effects memory of FDEs [52], application of FDEs for the modeling of anomalous diffusion [16], fractional operators models fluid flows of multiphase in porous media [15], fractional derivative viscoelastic dynamic models of solids mechanics [47], signal processing of fractional order [10] and applications of FC in engineering [30]. In 2022, Das et al. [9] have computed approximate solution to nonlinear integral equations and established the existence theory. In addition, Das and Rana [6] studied a class of fractional order weakly singular Volterra Integro differential equations by using convergence analysis. In 2020, Das et al. [8] used perturbation-based approach to compute approximate solution to integral equation. Das et al. [7] in 2019 have applied the homotopy perturbation method for computation of approximate solution to integral equations. Sumi et al. [41] studied coupled system of multiple scale reaction diffusion problems. Computational method for coupled boundary layer problems and Legendre spectral method have been used for numerical solutions, we refer to [34]. Sabawi et al. [39] have studied H^1 stability and convergence analysis of non-local weakly singular integro partial differential equations. Aakansha and Das [1] studied singularly perturbed Gierer-Meinhardt type nonlinear coupled systems of parabolic problems. Researchers have studied fractional order Pennes bioheat flow problem by using orthogonal polynomials [43] and Gurtin–Pipkin type integro-differential equations by using semi group theoretical approach [3]. Sarkar et al. [42] studied a semi-linear reaction-diffusion system networked by using convergence analysis. Kumar and Das [26] applied convergent analysis to multiple scale parabolic singularly perturbed convection-diffusion coupled systems. Recently researchers have used adaptive mesh based efficient method [27], quintic B–spline approximation method [5], mesh refinement method [45], second-order a priori and a posteriori error estimations [25] were used to study different problems.

The FC has an important property that it is defined in different ways to address some specific issues in the scientific fields. For, instance the concept of Riemann-Liouville is used mainly to study properties and problems of pure mathematics [33], the Caputo definition have been applied to some applied problems [36], the Caputo-Fabrizio and Atangana-Baleanu operators which have recently defined and used in problems which show memory effects and non-local behavior [48]. Morales-Delgado et al. [32] have studied the applications of Caputo-Fabrizio and Atangana-Baleanu fractional derivatives in cancer chemotherapy model. The CFDO is an important and very simple concept which has almost all important properties of classical derivative while the other fractional derivative do not have the properties like product rule, quotient rule, and chain rule [23]. This simple and fractional behavior of CFDO have been utilized by different scholar's and investigates some important problems of scientific disciplines. Here, we listed some significant application the CFDO like the dynamical behavior of tumor model [4], problems related to control systems [40], flow of fluid in porous media [28], and some important application in engineering [14]. Motivated by the applications of CFDO, researchers have used it extensively to address complex problems of real life.

As we know that systems of differential equations which are interdependent on each other form coupled systems. They have been extensively used for modeling of some inter dependent problems of different physical and biological fields. Such systems interconnect various changes of applied fields which depend on one another. The mentioned coupled system have been used in the past for the analysis of some important issues like, dynamical problems [12], problems of networks [37], ecological pre-predator coupled system [12]. Some significant problems in physics include modeling couple systems, such as chemical reactions [12], electric circuits [12], and mechanical oscillators [13] were studied by using tools of FC. Likewise, engineering challenges have been studied using the tools based on FC [20]. These systems, particularly in FC, are modeled to solve a lot of complicated issues that come up in our daily lives [49]. Ahmad and Alsaedi [2] have been studied existence theory of coupled system of integral fractional differential problems. Moreover, Gafiychuk et al. [17] have considered the area very well for analysis under different mathematical tools. On the other hand hybrid concept is an important when there is phenomenon depending on both continuous and discrete quantities such problems are very rarely consider for the qualitative and numerical study, we refer to [29]. Hence, FDEs in the form of coupled hybrid systems by using CFDO are considered.

The existence theory of functional equations is a broad concept and having very rich literature in the past. Before going to analyze a functional equation, particularly differential and integral equations it is crucial to see wether the problems are correctly model and also to check its validity and well-posedness. These queries have been addressed by the concept of mathematical analysis known is existence theory. Different strategies have been considered for such area of analysis including degree theory, functional analysis, fixed point analysis, Picard's existence theorems [22]. Among these methods the fixed point theory are very widely used because it cover most of the area of this topic with weak assumptions [19]. Another, concept of functional equations is stability which tells about under what circumstances the solution does not change much from its original position after very small change in he system [46]. Particularly, when we say that under what assumptions the exact solution must be closed to any approximate solution. These queries were answered by an important concepts of stability which was given by Ulam in 1940 [50]. The said concepts were further explored by Hyers [21] in 1941. In 1978, further the mentioned stability Ulam-Hyers (U-H) was generalized and analyzed by Rassias [38]. The mentioned stability have been explored by different mathematicians very openly applied to various functional equations for example see [44] and references there. But this concept of stability was very rarely investigated for the coupled systems particularly for hybrid coupled systems and need to address such problems for the mentioned stability [51].

Previously, researchers have used various semi analytical or numerical method to study specific problems for numerical and theoretical analysis. Our work is unique to study a general class of hybrid differential equations with nonlinear integral boundary conditions. Motivated by the discussion, in this article a hybrid coupled system of FDEs with non-local initial integral boundary conditions is under the study for qualitative and numerical analysis. The system is considered in the context of CFDO, such systems are not been studied too much for mentioned analysis under the proposed fractional operator. The hybrid system under our consideration for $\delta \in (0, 1]$ is formulated as follows:

$$\begin{cases} \mathcal{J}_0^{\delta}[\mathbf{x}(\varpi)-\mathbf{h_1}(\varpi,\mathbf{x}(\varpi))]=\mathbf{f_1}(\varpi,\mathbf{x}(\varpi),\mathbf{y}(\varpi)),\\ \mathcal{J}_0^{\delta}[\mathbf{y}(\varpi)-\mathbf{h_2}(\varpi,\mathbf{y}(\varpi))]=\mathbf{f_2}(\varpi,\mathbf{x}(\varpi),\mathbf{y}(\varpi)),\\ \quad \mathbf{x}(0)=\mathbf{x_0}+\displaystyle\int_0^{\tau}\mathbf{g_1}\left(\mu,\mathbf{x}(\mu)\right)\mathrm{d}\mu,\\ \quad \mathbf{y}(0)=\mathbf{y_0}+\displaystyle\int_0^{\tau}\mathbf{g_2}\left(\mu,\mathbf{y}(\mu)\right)\mathrm{d}\mu, \end{cases} \tag{1}$$

where, the continuous functions are defined by

$$\mathbf{f_1},\mathbf{f_2}\in C\left([0,\tau]\times\mathbf{R}^2\right),\ \mathbf{h_1},\mathbf{h_2},\mathbf{g_1},\mathbf{g_2}\in C\left([0,\tau]\times\mathbf{R}\right).$$

Further, the mentioned nonlinear functions in (1) satisfy the growth conditions and Lipschitz conditions which we have defined in Section 3. Here, it is obvious that the conditions involve nonlinear functions. Involving nonlinearity in initial conditions which can describe the abrupt and unforeseen changes in the system with long term behavior. Hence, this results will demonstrate a strong confections between the dynamics and initial conditions. Also, the qualitative theory is established by using fixed point theory and U-H concepts for stability analysis. Some numerical presentations are given by using RK-4 method to understand the dynamical interpretation.

Organization of article is follows as: Introduction is given in first section. Basic results are given in second section. The qualitative and numerical results are demonstrated in section 3 and section 4. The final outcomes are concluded in section 5.

2 Preliminaries

The following results are needed.

Definition 2.1. [23, 46] *Let $1\geq\delta>0$ and $\mathbf{x}\in C[0,\tau]$, then the CFDO can be described as:*

$$\mathcal{J}_0^{\delta}\mathbf{x}(\varpi)=\lim_{\mathrm{h}\to 0}\frac{\mathbf{x}(\varpi+\mathrm{h}\varpi^{1-\delta})-\mathbf{x}(\varpi)}{\mathrm{h}},\quad \delta\in(0,1].$$

Definition 2.2. [23, 46] *The corresponding integral for CFDO with $\delta>0$ can be described as:*

$$\mathbf{I}_0^{\delta}(\mathbf{x})(\varpi)=\int_0^{\varpi}\mu^{\delta-1}\mathbf{x}(\mu)\mathrm{d}\mu.$$

Lemma 2.1. [4, 40] *The given results holds:*

$$\mathcal{J}_0^{\delta}[\mathbf{I}_0^{\delta}(\mathbf{x})(\varpi)]=\mathbf{x}(\varpi).$$

Lemma 2.2. [4, 40] *The result given bellow hold:*

$$\mathbf{I}_0^{\delta}[\mathcal{J}_0^{\delta}(\mathbf{x}(\varpi))]=\mathbf{x}(\varpi)-\mathbf{x}(0).$$

If $\mathbf{W}=\mathrm{C}[0,\tau]$ be the Banach's spaces under the norm $\|\mathbf{x}\|=\sup_{\varpi\in[0,\tau]}\{|\mathbf{x}|:\mathbf{x}\in\mathbf{W}\}$. Additionally, the product space $\mathbf{W}\times\mathbf{W}$ endowed with norm $\|(\mathbf{x},\mathbf{y})\|=\|\mathbf{x}\|+\|\mathbf{y}\|$ constitutes a Banach space.

Theorem 2.1. [22] *A contraction mapping $\mathbf{Q}:\mathbf{W}\to\mathbf{W}$ has a unique fixed point.*

Theorem 2.2. [22] *If $\mathcal{D} \subset \mathbf{W}$, and there exists two operators $\mathbf{Q}_1$ and $\mathbf{Q}_2$ satisfying the following conditions:*

(1) $\mathbf{Q}_1(\mathbf{x}) + \mathbf{Q}_2(\mathbf{x}) \in \mathbf{W} \quad \forall \quad \mathbf{x} \in \mathbf{W}$.

(2) $\mathbf{Q}_1$ *represents contraction and the operator* $\mathbf{Q}_2$ *is completely continuous,*

$\mathbf{Q}_1(\mathbf{x}) + \mathbf{Q}_2(\mathbf{x}) = \mathbf{x}$ *has at least one fixed point.*

3 Qualitative Study

If $\mathbf{S} \subset \mathbf{W} \times \mathbf{W}$ of all bounded functions, we deduce the following results:

Lemma 3.1. *Consider the given linear coupled system for $\mathbf{k}_1, \mathbf{k}_2 : [0, \tau] \to \mathbf{R}$*

$$\begin{cases} \mathcal{J}_0^{\delta}\mathbf{x}(\varpi) = \mathbf{k}_1(\varpi), & \mathbf{x}(0) \quad = \mathbf{x_0}, \\ \mathcal{J}_0^{\delta}\mathbf{y}(\varpi) = \mathbf{k}_2(\varpi), & \mathbf{x}(0) \quad = \mathbf{x_0}, \end{cases} \tag{2}$$

which is equivalent to

$$\begin{cases} \mathbf{x}(\varpi) = \mathbf{x_0} + \displaystyle\int_0^{\varpi} \mu^{\delta-1}\mathbf{k}_1(\mu)\mathrm{d}\mu, \\ \mathbf{y}(\varpi) = \mathbf{y_0} + \displaystyle\int_0^{\varpi} \mu^{\delta-1}\mathbf{k}_2(\mu)\mathrm{d}\mu. \end{cases} \tag{3}$$

Proof. Using Lemma 2.2, system (2) yields

$$\begin{cases} \mathbf{x}(\varpi) - \mathbf{x}(0) = \mathbf{I}_0^{\delta}\mathbf{k}_1(\varpi), \\ \mathbf{y}(\varpi) - \mathbf{y}(0) = \mathbf{I}_0^{\delta}\mathbf{k}_2(\varpi). \end{cases} \tag{4}$$

Using initial conditions, one has from (4)

$$\begin{cases} \mathbf{x}(\varpi) = \mathbf{x_0} + \displaystyle\int_0^{\varpi} \mu^{\delta-1}\mathbf{k}_1(\mu)\mathrm{d}\mu, \\ \mathbf{y}(\varpi) = \mathbf{y_0} + \displaystyle\int_0^{\varpi} \mu^{\delta-1}\mathbf{k}_2(\mu)\mathrm{d}\mu. \end{cases} \tag{5}$$

□

From Equation (1),on using Lemma 3.1, we have

$$\begin{cases} \mathbf{x}(\varpi) = \mathbf{x_0} + \displaystyle\int_0^{\tau} \mathbf{g_1}(\mu, \mathbf{x}(\mu))\,\mathrm{d}\mu + \mathbf{h_1}(\varpi, \mathbf{x}(\varpi)) + \int_0^{\varpi} \mu^{\delta-1}\mathbf{f_1}(\mu, \mathbf{x}(\mu), \mathbf{y}(\mu))\mathrm{d}\mu, \\ \mathbf{y}(\varpi) = \mathbf{y_0} + \displaystyle\int_0^{\tau} \mathbf{g_2}(\mu, \mathbf{y}(\mu))\,\mathrm{d}\mu + \mathbf{h_2}(\varpi, \mathbf{y}(\varpi)) + \int_0^{\varpi} \mu^{\delta-1}\mathbf{f_2}(\mu, \mathbf{x}(\mu), \mathbf{y}(\mu))\mathrm{d}\mu. \end{cases} \tag{6}$$

Now, define some functions as $\mathbf{U}(\mathbf{x}, \mathbf{y}) : \mathbf{W} \times \mathbf{W} \to \mathbf{W} \times \mathbf{W}$, such that

$$\mathbf{U}(\mathbf{x}, \mathbf{y}) = \begin{cases} \mathbf{x_0} + \displaystyle\int_0^{\tau} \mathbf{g_1}(\mu, \mathbf{x}(\mu))\,\mathrm{d}\mu + \mathbf{h_1}(\varpi, \mathbf{x}(\varpi)) + \int_0^{\varpi} \mu^{\delta-1}\mathbf{f_1}(\mu, \mathbf{x}(\mu), \mathbf{y}(\mu))\mathrm{d}\mu, \\ \mathbf{y_0} + \displaystyle\int_0^{\tau} \mathbf{g_2}(\mu, \mathbf{y}(\mu))\,\mathrm{d}\mu + \mathbf{h_2}(\varpi, \mathbf{y}(\varpi)) + \int_0^{\varpi} \mu^{\delta-1}\mathbf{f_2}(\mu, \mathbf{x}(\mu), \mathbf{y}(\mu))\mathrm{d}\mu. \end{cases} \tag{7}$$

From Equation (7), we obtain a coupled fixed point problem as:

$$\mathbf{U}(\mathbf{x},\mathbf{y}) = (\mathbf{x},\mathbf{y}). \tag{8}$$

Here, we define $\mathbf{F} : \mathbf{W} \times \mathbf{W} \to \mathbf{W} \times \mathbf{W}$ by

$$\mathbf{F}(\mathbf{x},\mathbf{y}) = \begin{cases} \mathbf{x_0} + \displaystyle\int_0^{\tau} \mathbf{g_1}\left(\mu, \mathbf{x}(\mu)\right) \mathrm{d}\mu + \mathbf{h_1}\left(\varpi, \mathbf{x}(\varpi)\right), \\ \mathbf{y_0} + \displaystyle\int_0^{\tau} \mathbf{g_2}\left(\mu, \mathbf{y}(\mu)\right) \mathrm{d}\mu + \mathbf{h_2}\left(\varpi, \mathbf{y}(\varpi)\right). \end{cases} \tag{9}$$

and $\mathbb{G} : \mathbf{W} \times \mathbf{W} \to \mathbf{W} \times \mathbf{W}$,

$$\begin{cases} \mathbb{G}(\mathbf{x},\mathbf{y}) = \displaystyle\int_0^{\varpi} \mu^{\delta-1}\mathbf{f_1}(\mu, \mathbf{x}(\mu), \mathbf{y}(\mu))\mathrm{d}\mu, \\ \displaystyle\int_0^{\varpi} \mu^{\delta-1}\mathbf{f_2}(\mu, \mathbf{x}(\mu), \mathbf{y}(\mu))\mathrm{d}\mu. \end{cases} \tag{10}$$

Further, $\mathbf{F}(\mathbf{x},\mathbf{y}) = (\mathbf{F}_1(\mathbf{x}), \mathbf{F}_2(\mathbf{y}))$, where

$$\begin{cases} \mathbf{F}_1(\mathbf{x}) = \mathbf{x_0} + \displaystyle\int_0^{\tau} \mathbf{g_1}\left(\mu, \mathbf{x}(\mu)\right) \mathrm{d}\mu + \mathbf{h_1}\left(\varpi, \mathbf{x}(\varpi)\right). \\ \mathbf{F}_2(\mathbf{y}) = \mathbf{y_0} + \displaystyle\int_0^{\tau} \mathbf{g_2}\left(\mu, \mathbf{y}(\mu)\right) \mathrm{d}\mu + \mathbf{h_2}\left(\varpi, \mathbf{y}(\varpi)\right). \end{cases} \tag{11}$$

Describe

$$\mathbb{G}(\mathbf{x},\mathbf{y})(\varpi) = (\mathbb{G}_1, \mathbb{G}_2)(\mathbf{x},\mathbf{y})(\varpi),$$

where

$$\begin{cases} \mathbb{G}_1(\mathbf{x},\mathbf{y})(\varpi) = \displaystyle\int_0^{\varpi} \mu^{\delta-1}\mathbf{f_1}(\mu, \mathbf{x}(\mu), \mathbf{y}(\mu))\mathrm{d}\mu. \\ \mathbb{G}_2(\mathbf{x},\mathbf{y})(\varpi) = \displaystyle\int_0^{\varpi} \mu^{\delta-1}\mathbf{f_2}(\mu, \mathbf{x}(\mu), \mathbf{y}(\mu))\mathrm{d}\mu. \end{cases} \tag{12}$$

On the basis of Equation (9) and Equation (10), we have

$$(\mathbf{x},\mathbf{y}) = \mathbf{U}(\mathbf{x},\mathbf{y}) = \mathbf{F}(\mathbf{x},\mathbf{y}) + \mathbb{G}(\mathbf{x},\mathbf{y}). \tag{13}$$

The given hypothesis are needed:

$(\mathbf{A}_1)$ Corresponding to positive constants $\mathbf{a}$ and $\mathbf{b}$, we obtain

$$\begin{cases} |\mathbf{h_1}\left(\varpi, \mathbf{x}(\varpi)\right) - \mathbf{g_1}\left(\varpi, \bar{\mathbf{x}}(\varpi)\right)| \le \mathbf{a}|\mathbf{x}(\varpi) - \bar{\mathbf{x}}(\varpi)|, \\ |\mathbf{h_2}\left(\varpi, \mathbf{y}(\varpi)\right) - \mathbf{h_2}\left(\varpi, \bar{\mathbf{y}}(\varpi)\right)| \le \mathbf{b}|\mathbf{y}(\varpi) - \bar{\mathbf{y}}(\varpi)|. \end{cases} \tag{14}$$

$(\mathbf{A}_2)$ Against the positive real constants $\mathbf{a_1}$ and $\mathbf{b_1}$, we have

$$\begin{cases} |\mathbf{g_1}\left(\varpi, \mathbf{x}(\varpi)\right) - \mathbf{g_1}\left(\varpi, \bar{\mathbf{x}}\right)| \le \mathbf{a_1}|\mathbf{x}(\varpi) - \bar{\mathbf{x}}(\varpi)|, \\ |\mathbf{g_2}\left(\varpi, \mathbf{y}(\varpi)\right) - \mathbf{g_2}\left(\varpi, \bar{\mathbf{y}}(\varpi)\right)| \le \mathbf{b_1}|\mathbf{y}(\varpi) - \bar{\mathbf{y}}(\varpi)|. \end{cases} \tag{15}$$

$(\mathbf{A}_3)$ Let there exists positive real constants $\mathbf{a_2}, \mathbf{a_3}$, and $\mathbf{b_2}, \mathbf{b_3}$, then

$$\begin{cases} |\mathbf{f_1}\left(\varpi, \mathbf{x}, \mathbf{y}(\varpi)\right) - \mathbf{f_1}\left(\varpi, \bar{\mathbf{x}}(\varpi), \bar{\mathbf{y}}(\varpi)\right)| \le \mathbf{a_2}|\mathbf{x}(\varpi) - \bar{\mathbf{x}}(\varpi)| + \mathbf{a_3}|\mathbf{y}(\varpi) - \bar{\mathbf{y}}(\varpi)|, \\ |\mathbf{f_2}\left(\varpi, \mathbf{x}, \mathbf{y}\right) - \mathbf{f_2}\left(\varpi, \bar{\mathbf{x}}(\varpi), \bar{\mathbf{y}}(\varpi)\right)| \le \mathbf{b_2}|\mathbf{x}(\varpi) - \bar{\mathbf{x}}(\varpi)| + \mathbf{b_3}|\mathbf{y}(\varpi) - \bar{\mathbf{y}}(\varpi)|. \end{cases} \tag{16}$$

$(\mathbf{A}_4)$ If there exists positive real constants $\mathbf{a_4}, \mathbf{a_5}$, and $\mathbf{b_4}, \mathbf{b_5}$, then one has

$$\begin{cases} |\mathbf{f_1}(\varpi, \mathbf{x}(\varpi), \mathbf{y}(\varpi))| \le \mathbf{a_4}|\mathbf{x}(\varpi)| + \mathbf{a_5}|\mathbf{y}(\varpi)|, \\ |\mathbf{f_2}(\varpi, \mathbf{x}(\varpi), \mathbf{y}(\varpi))| \le \mathbf{b_4}|\mathbf{x}(\varpi)| + \mathbf{b_5}|\mathbf{y}(\varpi)|. \end{cases} \tag{17}$$

Theorem 3.1. *The considered coupled hybrid system* (1) *has at least one solution if* $\Lambda = \max\{(\mathbf{a_1}\tau + \mathbf{a}), (\mathbf{b_1}\tau + \mathbf{b})\} < 1$.

Proof. Consider $(\mathbf{x}, \mathbf{y}), (\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \mathbf{S} \subset \mathbf{W} \times \mathbf{W}$, where $\mathbf{S}$ is closed, convex, and bounded with $\|(\mathbf{x}, \mathbf{y})\| \le \mathbf{r}$, then

$$\begin{aligned}
\sup_{\varpi\in[0,\tau]} |\mathbf{F}(\mathbf{x}, \mathbf{y}) - \mathbf{F}(\bar{\mathbf{x}}, \bar{\mathbf{y}})| = & \sup_{\varpi\in\tau} \Bigg\{ \Bigg| \mathbf{x_0} + \int_0^\tau \mathbf{g_1}(\mu, \mathbf{x}(\mu))\, \mathrm{d}\mu + \mathbf{h_1}(\varpi, \mathbf{x}(\varpi)) \\
& + \mathbf{y_0} + \int_0^\tau \mathbf{g_2}(\mu, \mathbf{y}(\mu))\, \mathrm{d}\mu + \mathbf{h_2}(\varpi, \mathbf{y}(\varpi)) \\
& - \Bigg(\bar{\mathbf{x_0}} + \int_0^\tau \mathbf{g_1}(\mu, \bar{\mathbf{x}}(\mu))\, \mathrm{d}\mu + \mathbf{h_1}(\varpi, \bar{\mathbf{x}}(\varpi)) \\
& + \bar{\mathbf{y_0}} + \int_0^\tau \mathbf{g_2}(\mu, \bar{\mathbf{y}}(\mu))\, \mathrm{d}\mu + \mathbf{h_2}(\varpi, \bar{\mathbf{y}}(\varpi)) \, \} \Bigg) \Bigg| \Bigg\}, \\
\le & \sup_{\varpi\in[0,\tau]} \Bigg\{ \int_0^\tau |\mathbf{g_1}(\mu, \mathbf{x}(\mu)) - \mathbf{g_1}(\mu, \bar{\mathbf{x}}(\mu))| \mathrm{d}\mu \\
& + \int_0^\tau |\mathbf{g_2}(\mu, \mathbf{y}(\mu)) - \mathbf{g_2}(\mu, \bar{\mathbf{y}}(\mu))| \mathrm{d}\mu \\
& + |\mathbf{h_1}(\mu, \mathbf{x}(\mu)) - \mathbf{h_1}(\varpi, \bar{\mathbf{x}}(\varpi))| \\
& + |\mathbf{h_2}(\varpi, \mathbf{x}(\varpi)) - \mathbf{2_1}(\varpi, \bar{\mathbf{x}}(\varpi))| \} \}, \\
\le & \, \mathbf{a_1}\tau\|\mathbf{x} - \bar{\mathbf{x}}\| + \mathbf{b_1}\tau\|\mathbf{y} - \bar{\mathbf{y}}\| + \mathbf{a}\|\mathbf{x} - \bar{\mathbf{x}}\| + \mathbf{b}\|\mathbf{y} - \bar{\mathbf{y}}\|, \\
\le & \, (\mathbf{a_1}\tau + \mathbf{a})\|\mathbf{x} - \bar{\mathbf{x}}\| + (\mathbf{b_1}\tau + \mathbf{b})\|\mathbf{y} - \bar{\mathbf{y}}\|, \\
\le & \, \Lambda[\|(\mathbf{x}, \mathbf{y}) - (\bar{\mathbf{x}}, \bar{\mathbf{y}})\|], \text{ where } \Lambda = \max\{(\mathbf{a_1}\tau + \mathbf{a}), (\mathbf{b_1}\tau + \mathbf{b})\},
\end{aligned}$$

which implies, $\|\mathbf{F}(\mathbf{x}, \mathbf{y}) - \mathbf{F}(\bar{\mathbf{x}}, \bar{\mathbf{y}})\| \le \Lambda[\|(\mathbf{x}, \mathbf{y}) - (\bar{\mathbf{x}}, \bar{\mathbf{y}})\|]$. Hence, the function $\mathbf{F}$ is contraction.

Next, the function $\mathbb{G}$ is compact and continuous. Use $(\mathbf{x}, \mathbf{y}) \in \mathbf{S}$, one has

$$\begin{aligned}
\sup_{\varpi\in[0,\tau]} |\mathbb{G}(\mathbf{x}, \mathbf{y})| = & \sup_{\varpi\in[0,\tau]} \left\{ \left| \int_0^\varpi \mu^{\delta-1}\mathbf{f_1}(\mu, \mathbf{x}(\mu), \mathbf{y}(\mu))\mathrm{d}\mu + \int_0^\varpi \mu^{\delta-1}\mathbf{f_2}(\mu, \mathbf{x}(\mu), \mathbf{y}(\mu))\mathrm{d}\mu \right| \right\}, \\
\le & \sup_{\varpi\in[0,\tau]} \left\{ \int_0^\varpi \mu^{\delta-1}(|\mathbf{f_1}(\mu, \mathbf{x}(\mu), \mathbf{y}(\mu))| + |\mathbf{f_2}(\mu, \mathbf{x}(\mu), \mathbf{y}(\mu))|)\mathrm{d}\mu \right\}, \\
\le & \sup_{\varpi\in[0,\tau]} \left\{ \int_0^\varpi \mu^{\delta-1}(\mathbf{a_4}|\mathbf{x}(\mu)| + \mathbf{a_5}|\mathbf{y}(\mu)| + \mathbf{b_4}|\mathbf{y}(\mu)| + \mathbf{b_5}|\mathbf{x}(\mu)|)\mathrm{d}\mu \right\}, \\
\le & \, [\mathbf{a_4} + \mathbf{a_5} + \mathbf{b_4} + \mathbf{b_5}] \frac{\mathbf{r}\tau^\delta}{2\delta}, \\
< & \, \infty.
\end{aligned}$$

Hence, $\|\mathbb{G}\| < \infty$ so the map $\mathbb{G}$ is uniformly bounded and the continuity of $\mathbb{G}$ is insure from the continuity of $\mathbf{f_1}$ and $\mathbf{f_2}$. Now, for $\varpi_1 < \varpi_2$, we have

$$\begin{aligned}
&|\mathbb{G}(\mathbf{x}(\varpi_2),\mathbf{y}(\varpi_2)) - \mathbb{G}(\mathbf{x}(\varpi_1),\mathbf{y}(\varpi_1))| \\
&\qquad = \left|\int_0^{\varpi_2}\mu^{\delta-1}\mathbf{f_1}(\mu,\mathbf{x}(\mu),\mathbf{y}(\mu))\mathrm{d}\mu + \int_0^{\varpi_2}\mu^{\delta-1}\mathbf{f_2}(\mu,\mathbf{x}(\mu),\mathbf{y}(\mu))\mathrm{d}\mu\right| \\
&\qquad\quad - \left(\left|\int_0^{\varpi_1}\mu^{\delta-1}\mathbf{f_1}(\mu,\mathbf{x}(\mu),\mathbf{y}(\mu))\mathrm{d}\mu + \int_0^{\varpi_1}\mu^{\delta-1}\mathbf{f_2}(\mu,\mathbf{x}(\mu),\mathbf{y}(\mu))\mathrm{d}\mu\right|\right), \\
&\qquad \leq \int_0^{\varpi_2}\mu^{\delta-1}|\mathbf{f_1}(\mu,\mathbf{x}(\mu),\mathbf{y}(\mu))|\mathrm{d}\mu - \int_0^{\varpi_1}\mu^{\delta-1}|\mathbf{f_1}(\mu,\mathbf{x}(\mu),\mathbf{y}(\mu))|\mathrm{d}\mu \\
&\qquad\quad + (\int_0^{\varpi_2}\mu^{\delta-1}|\mathbf{f_2}(\mu,\mathbf{x}(\mu),\mathbf{y}(\mu))|\mathrm{d}\mu - \int_0^{\varpi_1}\mu^{\delta-1}|\mathbf{f_2}(\mu,\mathbf{x}(\mu),\mathbf{y}(\mu))|\mathrm{d}\mu, \\
&\qquad \leq (\mathbf{a_4}+\mathbf{a_5})\frac{\mathbf{r}}{2\delta}\times\left[\varpi_2^\delta - \varpi_1^\delta\right] + (\mathbf{b_4}+\mathbf{b_5})\frac{\mathbf{r}}{2\delta}\times\left[\varpi_2^\delta - \varpi_1^\delta\right], \\
&\qquad \leq \left[\mathbf{a_4}+\mathbf{a_5}+\mathbf{b_4}+\mathbf{b_5}\right]\frac{\mathbf{r}}{2\delta}\times\left[\varpi_2^\delta - \varpi_1^\delta\right] \to 0 \text{ as } \varpi_1 \to \varpi_2.
\end{aligned}$$

Since all conditions of Theorem 2.2 are satisfied, therefore the system (1) has at least one solution. □

Theorem 3.2. *The proposed coupled system* (1) *has a unique solution if*

$$\Delta = \max\{[\mathbf{a_1}\tau + \mathbf{a} + \mathbf{a_2} + \mathbf{b_2}], [\mathbf{b_1}\tau + \mathbf{b} + \mathbf{a_3} + \mathbf{b_3}]\} < 1.$$

Proof. By taking $(\mathbf{x},\bar{\mathbf{x}}) \in \mathbf{W}\times\mathbf{W}$, we have

$$\begin{aligned}
|\mathbf{U}(\mathbf{x},\mathbf{y}) - \mathbf{U}(\bar{\mathbf{x}},\bar{\mathbf{y}})| &= \Bigg|\mathbf{x_0} + \int_0^\tau \mathbf{g_1}(\mu,\mathbf{x}(\mu))\,\mathrm{d}\mu + \mathbf{h_1}(\varpi,\mathbf{x}(\varpi)) + \mathbf{y_0} \\
&\quad + \int_0^\tau \mathbf{g_2}(\mu,\mathbf{y}(\mu))\,\mathrm{d}\mu + \mathbf{h_2}(\varpi,\mathbf{y}(\varpi)) \\
&\quad + \int_0^\varpi \mu^{\delta-1}\mathbf{f_1}(\mu,\mathbf{x}(\mu),\mathbf{y}(\mu))\mathrm{d}\mu + \int_0^\varpi \mu^{\delta-1}\mathbf{f_2}(\mu,\mathbf{x}(\mu),\mathbf{y}(\mu))\mathrm{d}\mu \\
&\quad - \Bigg(\Bigg|\bar{\mathbf{x}}_\mathbf{0} + \int_0^\tau \mathbf{g_1}(\mu,\bar{\mathbf{x}}(\mu))\,\mathrm{d}\mu + \mathbf{h_1}(\varpi,\bar{\mathbf{x}}(\varpi)) + \bar{\mathbf{y}}_\mathbf{0} \\
&\quad + \int_0^\tau \mathbf{g_2}(\mu,\bar{\mathbf{y}}(\mu))\,\mathrm{d}\mu + \mathbf{h_2}(\varpi,\bar{\mathbf{y}}(\varpi)) \\
&\quad + \int_0^\varpi \mu^{\delta-1}\mathbf{f_1}(\mu,\bar{\mathbf{x}}(\mu),\bar{\mathbf{y}}(\mu))\mathrm{d}\mu + \int_0^\varpi \mu^{\delta-1}\mathbf{f_2}(\mu,\bar{\mathbf{x}}(\mu),\bar{\mathbf{y}}(\mu))\mathrm{d}\mu\Bigg)\Bigg|, \\
&\leq \int_0^\tau |\mathbf{g_1}(\mu,\mathbf{x}(\mu)) - \mathbf{g_1}(\mu,\bar{\mathbf{x}}(\mu))|\,\mathrm{d}\mu + \int_0^\tau |\mathbf{g_2}(\mu,\mathbf{y}(\mu)) - \mathbf{g_2}(\mu,\bar{\mathbf{y}}(\mu))|\,\mathrm{d}\mu \\
&\quad + |\mathbf{h_1}(\varpi,\mathbf{x}(\varpi)) - \mathbf{h_1}(\varpi,\bar{\mathbf{x}}(\varpi))| + |\mathbf{h_2}(\varpi,\mathbf{y}(\varpi)) - \mathbf{h_2}(\varpi,\bar{\mathbf{y}}(\varpi))| \\
&\quad + \int_0^\varpi \mu^{\delta-1}\,|\mathbf{f_1}(\mu,\mathbf{x}(\mu),\mathbf{y}(\mu)) - \mathbf{f_1}(\mu,\bar{\mathbf{x}}(\mu),\bar{\mathbf{y}}(\mu))|\,\mathrm{d}\mu \\
&\quad + \int_0^\varpi \mu^{\delta-1}\,|\mathbf{f_2}(\mu,\mathbf{x}(\mu),\mathbf{y}(\mu)) - \mathbf{f_2}(\mu,\bar{\mathbf{x}}(\mu),\bar{\mathbf{y}}(\mu))|\,\mathrm{d}\mu, \\
&\leq (\mathbf{a_1}\tau)\|\mathbf{x}-\bar{\mathbf{x}}\| + (\mathbf{b_1}\tau)\|\mathbf{y}-\bar{\mathbf{y}}\| + \mathbf{a}\|\mathbf{x}-\bar{\mathbf{x}}\| + \mathbf{b}\|\mathbf{y}-\bar{\mathbf{y}}\| \\
&\quad + \mathbf{a_2}\|\mathbf{x}-\bar{\mathbf{x}}\| + \mathbf{a_3}\|\mathbf{y}-\bar{\mathbf{y}}\| + \mathbf{b_2}\|\mathbf{x}-\bar{\mathbf{x}}\| + \mathbf{b_3}\|\mathbf{y}-\bar{\mathbf{y}}\|, \\
&= [\mathbf{a_1}\tau + \mathbf{a} + \mathbf{a_2} + \mathbf{b_2}]\|\mathbf{x}-\bar{\mathbf{x}}\| + [\mathbf{b_1}\tau + \mathbf{b} + \mathbf{a_3} + \mathbf{b_3}]\|\mathbf{y}-\bar{\mathbf{y}}\|, \\
&\leq \Delta\|(\mathbf{x},\mathbf{y}) - (\bar{\mathbf{x}},\bar{\mathbf{y}})\|.
\end{aligned}$$

Thus, system (1) has a unique solution. □

3.1 Theory of Stability

For U-H stability results, let there exist two mappings $\mathbf{v}, \mathbf{s}$ independent of $\mathbf{x}, \mathbf{y}$ respectively such that for any $\epsilon > 0$

$$\mathbf{v}(\varpi) < \frac{\epsilon}{2}, \quad \mathbf{s}(\varpi) < \frac{\epsilon}{2}.$$

We can express coupled hybrid system of (1) with perturbation functions as:

$$\begin{cases} \mathcal{J}_0^{\delta}[\mathbf{x}(\varpi) - \mathbf{h_1}(\varpi, \mathbf{x})] = \mathbf{f_1}(\varpi, \mathbf{x}(\varpi), \mathbf{y}(\varpi)) + \mathbf{v}(\varpi), \\ \mathcal{J}_0^{\delta}[\mathbf{y}(\varpi) - \mathbf{h_2}(\varpi, \mathbf{y})] = \mathbf{f_2}(\varpi, \mathbf{x}(\varpi), \mathbf{y}(\varpi)) + \mathbf{v}(\varpi), \\ \quad \mathbf{x}(0) = \mathbf{x_0} + \displaystyle\int_0^{\tau} \mathbf{g_1}(\mu, \mathbf{x}(\mu))\, \mathrm{d}\mu, \\ \quad \mathbf{y}(0) = \mathbf{y_0} + \displaystyle\int_0^{\tau} \mathbf{g_2}(\mu, \mathbf{y}(\mu))\, \mathrm{d}\mu, \end{cases} \tag{18}$$

then, we have

$$|\mathbf{U}(\mathbf{x}(\varpi), \mathbf{y}(\varpi)) - (\mathbf{x}(\varpi), \mathbf{y}(\varpi))| \leq \frac{\tau^{\delta}\epsilon}{\delta}. \tag{19}$$

Lemma 3.2. *Proof.* Applying fractional integration and after using Theorem 3.2, the Equation (18) implies

$$\begin{aligned} (\mathbf{x}(\varpi), \mathbf{y}(\varpi)) &= \begin{cases} \mathbf{x_0} + \displaystyle\int_0^{\tau} \mathbf{g_1}(\mu, \mathbf{x}(\mu))\, \mathrm{d}\mu + \mathbf{h_1}(\mu, \mathbf{x}(\mu)) + \int_0^{\varpi} \mu^{\delta-1} \mathbf{f_1}(\mu, \mathbf{x}(\mu), \mathbf{y}(\mu)) \mathrm{d}\mu, \\ + \displaystyle\int_0^{\varpi} \mu^{\delta-1} \mathbf{v}(\mu) \mathrm{d}\mu + \mathbf{y_0} + \int_0^{\tau} \mathbf{g_2}(\mu, \mathbf{y}(\mu))\, \mathrm{d}\mu + \mathbf{h_2}(\mu, \mathbf{y}(\mu)), \\ + \displaystyle\int_0^{\varpi} \mu^{\delta-1} \mathbf{f_2}(\mu, \mathbf{x}(\mu), \mathbf{y}(\mu))\, \mathrm{d}\mu + \int_0^{\varpi} \mu^{\delta-1} \mathbf{s}(\mu) \mathrm{d}\mu, \end{cases} \\ &= \mathbf{U}(\mathbf{x}(\varpi), \mathbf{y}(\varpi)) + \int_0^{\varpi} \mu^{\delta-1} \mathbf{v}(\mu) \mathrm{d}\mu + \int_0^{\varpi} \mu^{\delta-1} \mathbf{s}(\mu) \mathrm{d}\mu. \end{aligned}$$

Therefore

$$\begin{aligned} |(\mathbf{x}(\varpi), \mathbf{y}(\varpi)) - \mathbf{U}(\mathbf{x}(\varpi), \mathbf{y}(\varpi))| &\leq \int_0^{\varpi} \mu^{\delta-1} |\mathbf{v}(\mu)| \mathrm{d}\mu + \int_0^{\varpi} \mu^{\delta-1} |\mathbf{s}(\mu)| \mathrm{d}\mu, \\ &\leq [\frac{\epsilon}{2} + \frac{\epsilon}{2}] \frac{\tau^{\delta}}{\delta}, \\ &= \frac{\tau^{\delta}\epsilon}{\delta}. \end{aligned}$$

□

Theorem 3.3. *The solution of hybrid coupled system (1) is stable in the sense of U-H if $\Delta < 1$.*

Proof. If $(\bar{\mathbf{x}}(\varpi), \bar{\mathbf{y}}(\varpi)) \in \mathbf{W} \times \mathbf{W}$ be any solution of system (18) and $(\mathbf{x}(\varpi), \mathbf{y}(\varpi)) \in \mathbf{W} \times \mathbf{W}$ is unique solutions of (1), then one has

$$\begin{aligned}
|(\mathbf{x}(\varpi),\mathbf{y}(\varpi))-(\bar{\mathbf{x}}(\varpi),\bar{\mathbf{y}}(\varpi))| &= |(\mathbf{x}(\varpi),\mathbf{y}(\varpi))-\mathbf{U}(\bar{\mathbf{x}}(\varpi),\bar{\mathbf{y}}(\varpi))|,\\
&= |(\mathbf{x}(\varpi),\mathbf{y}(\varpi))-\mathbf{U}(\mathbf{x}(\varpi),\mathbf{y}(\varpi)),\\
&\quad +\mathbf{U}(\mathbf{x}(\varpi),\mathbf{y}(\varpi))-\mathbf{U}(\bar{\mathbf{x}}(\varpi),\bar{\mathbf{y}}(\varpi))|,\\
&\le |(\mathbf{x}(\varpi),\mathbf{y}(\varpi))-\mathbf{U}(\mathbf{x}(\varpi),\mathbf{y}(\varpi))|\\
&\quad +|\mathbf{U}(\mathbf{x}(\varpi),\mathbf{y}(\varpi))-\mathbf{U}(\bar{\mathbf{x}}(\varpi),\bar{\mathbf{y}}(\varpi))|,\\
&\le \frac{\tau^{\delta}\epsilon}{\delta}+\Delta|(\mathbf{x}(\varpi),\mathbf{y}(\varpi))-(\bar{\mathbf{x}}(\varpi),\bar{\mathbf{y}}(\varpi))|,\\
&\le \frac{\tau^{\delta}}{(1-\Delta)\delta}\epsilon.
\end{aligned}$$

Hence, system (1) is U-H type stable. □

4 Numerical Solution

Lastly, the approximate solution of the hybrid coupled system (1) is examined in this section using the fractional RK-4 approach [31]. The interval $[0,\tau]$ is divided into sub-intervals as $[\varpi_l,\varpi_{l+1}]$ and $t_{\mathbf{q}+1}=t_{\mathbf{q}}+h$, where $h=\frac{\tau}{\mathbf{q}}$ on the interval $[0,\tau]$, we deduce the required method as follows:

$$\begin{cases}\mathcal{J}_0^{\delta}\mathbf{z}(\varpi)=\mathbf{F}(\varpi,\mathbf{z}(\varpi)), \quad \varpi\in[0,\tau], \quad 0<\delta\le 1,\\ \quad \mathbf{z}(0)=\mathbf{z}_0,\end{cases} \tag{20}$$

subject to CFDO as

$$\mathbf{z}^{m+1}=\mathbf{z}^{m}+\frac{h^{\delta}}{6\delta}[\mathbf{k}_1+2(\mathbf{k}_2+\mathbf{k}_3)+\mathbf{k}_4], \text{ for } m=0,1,2,3,\dots, \tag{21}$$

while

$$\begin{aligned}
\mathbf{k}_1&=\mathbf{F}\left(\varpi^{m},\mathbf{z}^{m}\right),\\
\mathbf{k}_2&=\mathbf{F}\left(\varpi^{m}+\frac{h^{\delta}}{2\delta},\mathbf{z}^{m}+\frac{h^{\delta}}{2\delta}\mathbf{k}_1\right),\\
\mathbf{k}_3&=\mathbf{F}\left(\varpi^{m}+\frac{h^{\delta}}{2\delta},\mathbf{z}^{m}+\frac{h^{\delta}}{2\delta}\mathbf{k}_2\right),\\
\mathbf{k}_4&=\mathbf{F}\left(\varpi^{m}+\frac{h^{\delta}}{\delta},\mathbf{z}^{m}+\frac{h^{\delta}}{\delta}\mathbf{k}_3\right).
\end{aligned}$$

Corresponding to Equation (21), the numerical solution of system (1) can be obtained as:

$$\begin{cases}\mathbf{x}^{m+1}=\mathbf{x}^{m}+\dfrac{1}{h}\displaystyle\sum_{l=0}^{\mathbf{q}}\left[\mathbf{g_1}\left(\varpi^{l+1},\mathbf{x}^{l+1}\right)-\mathbf{g_1}\left(\varpi^{l},\mathbf{x}^{l}\right)\right]+\mathbf{h_1}\left(\varpi^{m},\mathbf{x}^{m}\right)\\
\qquad +\dfrac{h^{\delta}}{6\delta}[\mathbf{k}_1+2(\mathbf{k}_2+\mathbf{k}_3)+\mathbf{k}_4], \quad m=0,1,2,3,\dots,\\
\mathbf{y}^{m+1}=\mathbf{y}^{m}+\dfrac{1}{h}\displaystyle\sum_{l=0}^{\mathbf{q}}\left[\mathbf{g_2}\left(\varpi^{l+1},\mathbf{y}^{l+1}\right)-\mathbf{g_2}\left(\varpi^{l},\mathbf{y}^{l}\right)\right]+\mathbf{h_2}\left(\varpi^{m},\mathbf{y}^{m}\right)\\
\qquad +\dfrac{h^{\delta}}{6\delta}[\mathbf{p}_1+2(\mathbf{p}_2+\mathbf{p}_3)+\mathbf{p}_4], \quad m=0,1,2,3,\dots,\end{cases} \tag{22}$$

such that

$$\begin{aligned}
\mathbf{k}_1 &= \mathbf{f_1}(\varpi^m, \mathbf{x}^m, \mathbf{y}^m),\\
\mathbf{k}_2 &= \mathbf{f_1}(\varpi^m + \frac{h^\delta}{2\delta}, \mathbf{x}^m + \frac{h^\delta}{2\delta}\mathbf{k}_1, \mathbf{y}^m + \frac{h^\delta}{2\delta}\mathbf{k}_1),\\
\mathbf{k}_3 &= \mathbf{f_1}(\varpi^m + \frac{h^\delta}{2\delta}, \mathbf{x}^m + \frac{h^\delta}{2\delta}\mathbf{k}_2, \mathbf{y}^m + \frac{h^\delta}{2\delta}\mathbf{k}_2),\\
\mathbf{k}_4 &= \mathbf{f_1}(\varpi^m + \frac{h^\delta}{\delta}, \mathbf{x}^m + \frac{h^\delta}{\delta}\mathbf{k}_3, \mathbf{y}^m + h^\delta\mathbf{k}_3)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{p}_1 &= \mathbf{f_2}(\varpi^m, \mathbf{x}^m, \mathbf{y}^m),\\
\mathbf{p}_2 &= \mathbf{f_2}(\varpi^m + \frac{h^\delta}{2\delta}, \mathbf{x}^m + \frac{h^\delta}{2\delta}\mathbf{p}_1, \mathbf{y}^m + \frac{h^\delta}{2\delta}\mathbf{p}_1),\\
\mathbf{p}_3 &= \mathbf{f_2}(\varpi^m + \frac{h^\delta}{2\delta}, \mathbf{x}^m + \frac{h^\delta}{2\delta}\mathbf{p}_2, \mathbf{y}^m + \frac{h^\delta}{2\delta}\mathbf{p}_2),\\
\mathbf{p}_4 &= \mathbf{f_2}(\varpi^m + \frac{h^\delta}{\delta}, \mathbf{x}^m + h^\delta\mathbf{p}_3, \mathbf{y}^m + \frac{h^\delta}{\delta}\mathbf{p}_3).
\end{aligned}$$

The convergence, the traditional method of RK-4 has a convergence error is of $O(h^4)$, where h is step size. But in case of fractional order a specialized formula is established whose convergence rate depends on the fractional order. For some ideal cases it may be $O(h^{4\delta})$. We refer [18] for detail convergence analysis of fractional order RK-4 method.

4.1 Applications

Example 4.1. *Here, a radioactive decay model of two species [12] is considered to apply our presented results. The system is described as:*

$$\begin{cases}
\mathcal{J}_0^\delta \mathbf{x}(\varpi) = -\lambda_1 \mathbf{x}(\varpi),\\
\mathcal{J}_0^\delta \mathbf{y}(\varpi) = \lambda_1 \mathbf{x}(\varpi) - \lambda_2 \mathbf{y}(\varpi),\\
\quad \mathbf{x}(0) = 1,\\
\quad \mathbf{y}(0) = 0.
\end{cases} \tag{23}$$

The constant λ_1 shows the rate of decays of the specie $\mathbf{x}$ into $\mathbf{y}$ and the constant λ_2 represents the rate of decays of the specie $\mathbf{y}$ into another form. The exact solution of differential equation $\mathcal{J}_0^\delta \mathbf{x}(\varpi) + \lambda \mathbf{x}(\varpi) = 0$ given in [24] is expressed as

$$\mathbf{x}(\varpi) = C\exp(-\lambda \frac{\varpi^\delta}{\delta}). \tag{24}$$

Using the above idea the exact solution of our first equation of the proposed system is given by

$$\mathbf{x}(\varpi) = C_1 \exp(-\lambda_1 \frac{\varpi^\delta}{\delta}), \tag{25}$$

Using Equation (25) and integrating factor $\exp(\lambda_2 \frac{\varpi^\delta}{\delta})$ the exact solution of the second equation of system (23) is formulated as follows,

$$\mathbf{y}(\varpi) = C_1 \frac{\lambda_1}{\lambda_2 - \lambda_1} \exp(-\lambda_1 \frac{\varpi^\delta}{\delta}) + C_2 \exp(-\lambda_2 \frac{\varpi^\delta}{\delta}), \ \textit{where } \lambda_2 \neq \lambda_1. \tag{26}$$

Thus, the general solution of our considered system is given by

$$\begin{cases} \mathbf{x}(\varpi) = C_1 \exp(-\lambda_1 \dfrac{\varpi^\delta}{\delta}), \\ \mathbf{y}(\varpi) = C_1 \dfrac{\lambda_1}{\lambda_2 - \lambda_1} \exp(-\lambda_1 \dfrac{\varpi^\delta}{\delta}) + C_2 \exp(-\lambda_2 \dfrac{\varpi^\delta}{\delta}), \text{ where } \lambda_2 \neq \lambda_1. \end{cases} \tag{27}$$

Applying the initial conditions $\mathbf{x}(0) = 1$ *and* $\mathbf{y}(0) = 0$*, the value of constants are* $C_1 = 1$ *and* $C_2 = -\frac{\lambda_1}{\lambda_2 - \lambda_1}$. *Hence, the particular solution of our considered system is given by*

$$\begin{cases} \mathbf{x}(\varpi) = \exp(-\lambda_1 \dfrac{\varpi^\delta}{\delta}), \\ \mathbf{y}(\varpi) = \dfrac{\lambda_1}{\lambda_2 - \lambda_1} \left(\exp(-\lambda_1 \dfrac{\varpi^\delta}{\delta}) - \exp(-\lambda_2 \dfrac{\varpi^\delta}{\delta}) \right), \text{ where } \lambda_2 \neq \lambda_1. \end{cases} \tag{28}$$

In Figures, 1-4, the exact solution of coupled system (23) for different fractional orders are presented.

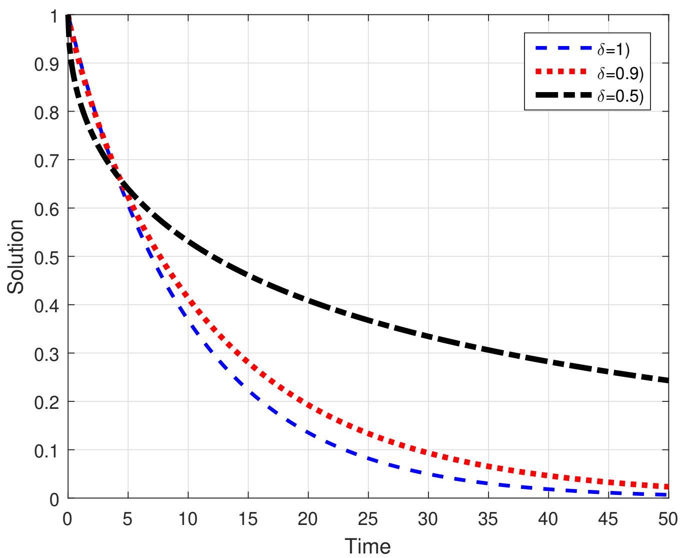

Figure 1: The exact solution of compartment $\mathbf{x}$ of coupled system (23).

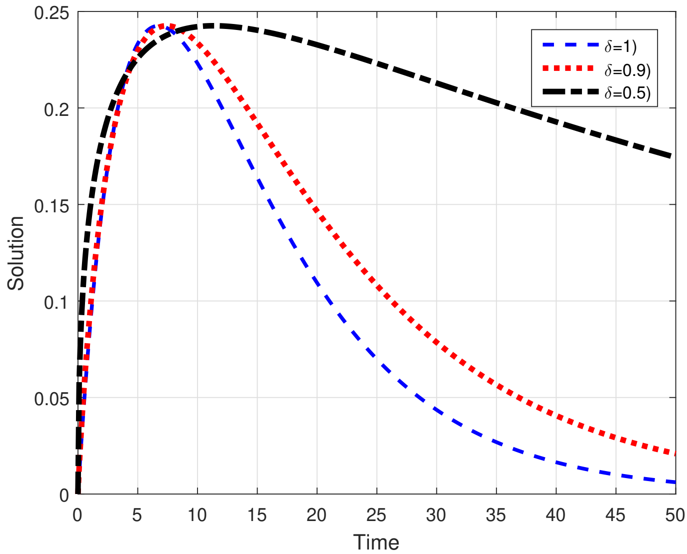

Figure 2: The exact solution of compartment $\mathbf{y}$ of coupled system (23).

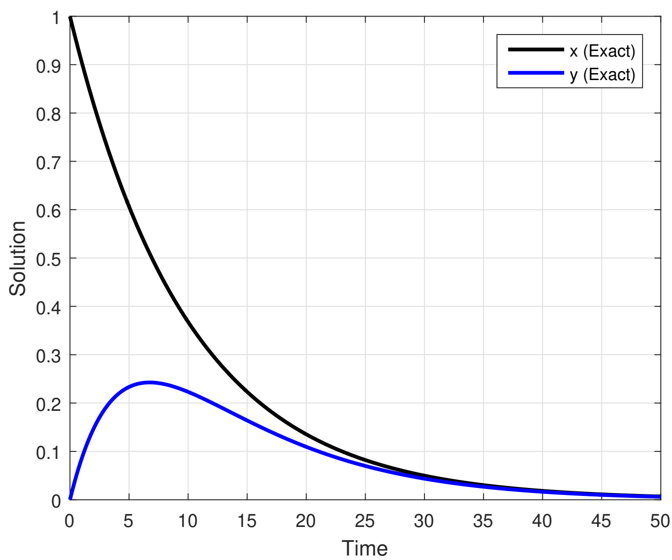

Figure 3: The exact solution coupled system (23).

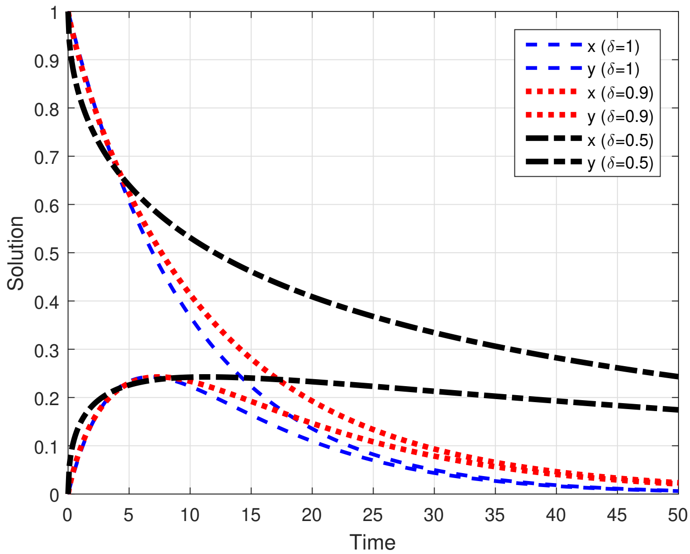

Figure 4: The exact solution coupled system (23).

Now, here we applied our analysis to the system (23). The corresponding functions of model (1) are defined as:

$$\begin{aligned}
\mathbf{f_1}(\varpi, \mathbf{x}(\varpi), \mathbf{y}(\varpi)) &= -\lambda_1 \mathbf{x}(\varpi),\\
\mathbf{f_2}(\varpi, \mathbf{x}(\varpi), \mathbf{y}(\varpi)) &= \lambda_1 \mathbf{x}(\varpi) - \lambda_2 \mathbf{y}(\varpi),\\
\mathbf{h_1}(\varpi, \mathbf{x}(\varpi)) &= 0,\\
\mathbf{h_2}(\varpi, \mathbf{y}(\varpi)) &= 0,\\
\mathbf{g_1}(\varpi, \mathbf{x}(\varpi)) &= 0,\\
\mathbf{g_2}(\varpi, \mathbf{y}(\varpi)) &= 0.
\end{aligned}$$

The assumptions of Theorems 3.1, 3.2 and 3.3 are easy to deduce for multiple values of the parameters λ_1 and λ_2. Hence, the system has a unique solution, and stable in the sense of U-H concept. Further, the numerical scheme developed in (22) has been applied. The solution of the system (23) for different fractional order has also displayed. The approximate solution mathematically formulated as follows:

$$\begin{cases}
\mathbf{x}^{m+1} = \mathbf{x}^m + \dfrac{h^\delta}{6\delta}[\mathbf{k}_1 + 2(\mathbf{k}_2 + \mathbf{k}_3) + \mathbf{k}_4], \quad m = 0, 1, 2, 3, \ldots,\\
\mathbf{y}^{m+1} = \mathbf{y}^m + \dfrac{h^\delta}{6\delta}[\mathbf{p}_1 + 2(\mathbf{p}_2 + \mathbf{p}_3) + \mathbf{p}_4], \quad m = 0, 1, 2, 3, \ldots,
\end{cases} \tag{29}$$

where

$$
\begin{aligned}
\mathbf{k}_1 &= \mathbf{f_1}(\varpi^m, \mathbf{x}^m, \mathbf{y}^m),\\
&= -\lambda_1 \mathbf{x}^m,\\
\mathbf{k}_2 &= \mathbf{f_1}(\varpi^m + \frac{h^\delta}{2\delta}, \mathbf{x}^m + \frac{h^\delta}{2\delta}\mathbf{k}_1, \mathbf{y}^m + \frac{h^\delta}{2\delta}\mathbf{k}_1),\\
&= -\lambda_1 \left[\mathbf{x}^m + \frac{h^\delta}{2\delta}\mathbf{k}_1\right],\\
\mathbf{k}_3 &= \mathbf{f_1}(\varpi^m + \frac{h^\delta}{2\delta}, \mathbf{x}^m + \frac{h^\delta}{2\delta}\mathbf{k}_2, \mathbf{y}^m + \frac{h^\delta}{2\delta}\mathbf{k}_2),\\
&= -\lambda_1 \left[\mathbf{x}^m + \frac{h^\delta}{2\delta}\mathbf{k}_2\right],\\
\mathbf{k}_4 &= \mathbf{f_1}(\varpi^m + \frac{h^\delta}{\delta}, \mathbf{x}^m + \frac{h^\delta}{\delta}\mathbf{k}_3, \mathbf{y}^m + h^\delta\mathbf{k}_3),\\
&= -\lambda_1 \left[\mathbf{x}^m + \frac{h^\delta}{\delta}\mathbf{k}_3\right].
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{p}_1 &= \mathbf{f_2}(\varpi^m, \mathbf{x}^m, \mathbf{y}^m),\\
&= \lambda_1 \mathbf{x}^m - \lambda_2 \mathbf{y}^m,\\
\mathbf{p}_2 &= \mathbf{f_2}(\varpi^m + \frac{h^\delta}{2\delta}, \mathbf{x}^m + \frac{h^\delta}{2\delta}\mathbf{p}_1, \mathbf{y}^m + \frac{h^\delta}{2\delta}\mathbf{p}_1),\\
&= \lambda_1 \left[\mathbf{x}^m + \frac{h^\delta}{2\delta}\mathbf{p}_1\right] - \lambda_2 \left[\mathbf{y}^m + \frac{h^\delta}{2\delta}\mathbf{p}_1\right],\\
\mathbf{p}_3 &= \mathbf{f_2}(\varpi^m + \frac{h^\delta}{2\delta}, \mathbf{x}^m + \frac{h^\delta}{2\delta}\mathbf{p}_2, \mathbf{y}^m + \frac{h^\delta}{2\delta}\mathbf{p}_2),\\
&= \lambda_1 \left[\mathbf{x}^m + \frac{h^\delta}{2\delta}\mathbf{p}_2\right] - \lambda_2 \left[\mathbf{y}^m + \frac{h^\delta}{2\delta}\mathbf{p}_2\right],\\
\mathbf{p}_4 &= \mathbf{f_2}(\varpi^m + \frac{h^\delta}{\delta}, \mathbf{x}^m + h^\delta\mathbf{p}_3, \mathbf{y}^m + \frac{h^\delta}{\delta}\mathbf{p}_3),\\
&= \lambda_1 \left[\mathbf{x}^m + \frac{h^\delta}{\delta}\mathbf{p}_3\right] - \lambda_2 \left[\mathbf{y}^m + \frac{h^\delta}{\delta}\mathbf{p}_3\right].
\end{aligned}
$$

In Figures 5-7, the numerical results of (23) for various fractional orders by using $h = 0.1$ and $\lambda_1 = 0.1, \lambda_2 = 0.21$ are demonstrated.

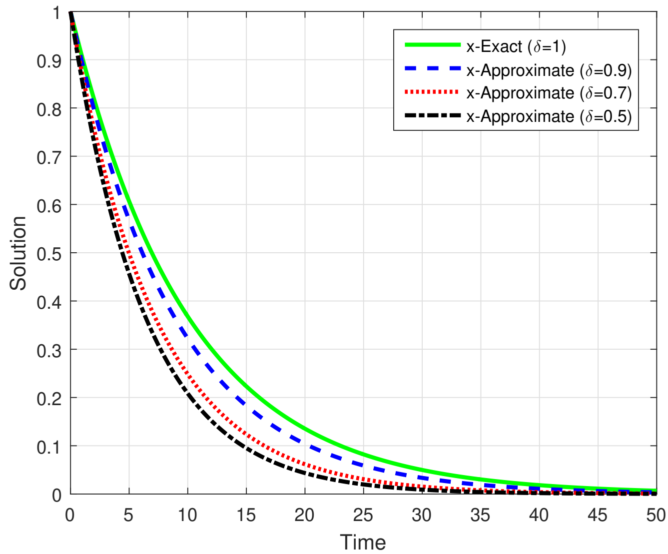

Figure 5: The approximate solution of compartment $\mathbf{x}$ of system (23).

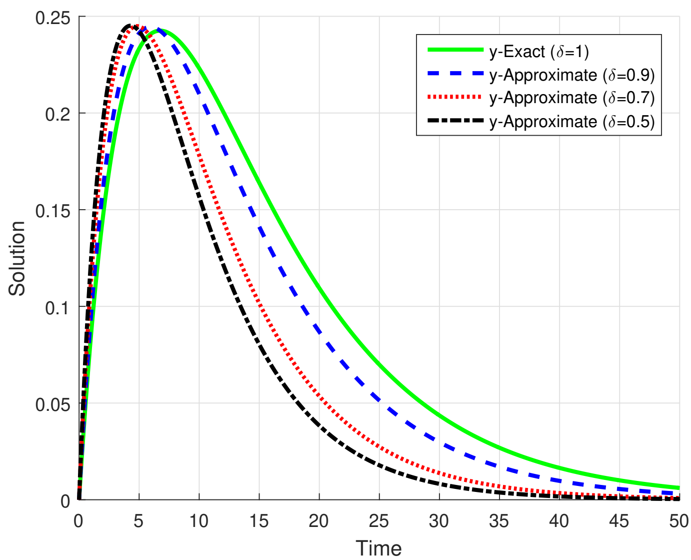

Figure 6: The approximate solution of compartment $\mathbf{y}$ of system (23).

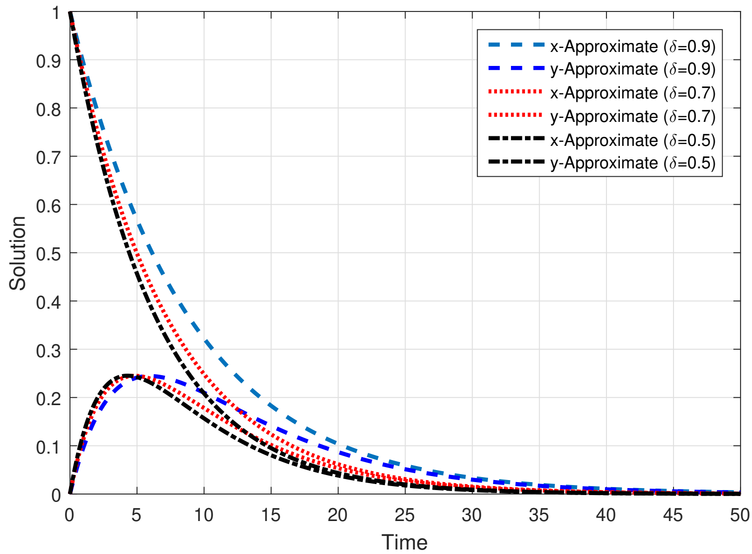

Figure 7: The approximate solution of coupled system (23).

By adding hybrid terms on the right under the fractional operator and with non-local conditions, the coupled system (23) *is generalized. The definition of the generalized coupled system that corresponds to system* (23) *is*

$$\begin{cases} \mathcal{J}_0^{\delta}[\mathbf{x}(\varpi) - \alpha_1 \exp(-\mathbf{x}(\varpi))] = -\lambda_1 \mathbf{x}(\varpi), \\ \mathcal{J}_0^{\delta}[\mathbf{y}(\varpi) - \alpha_2 \exp(-\mathbf{y}(\varpi))] = \lambda_1 \mathbf{x}(\varpi) - \lambda_2 \mathbf{y}(\varpi), \\ \mathbf{x}(0) = 1 + \beta_1 \displaystyle\int_0^{\tau} \sin(\mathbf{x}(\mu))\mathrm{d}\mu, \\ \mathbf{y}(0) = \beta_2 \displaystyle\int_0^{\tau} \cos(\mathbf{y}(\mu))\mathrm{d}\mu. \end{cases} \tag{30}$$

Here, $\delta \in (0,1]$, $\varpi \in [0,\tau]$ *and* $\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda_1, \lambda_2$, *are constants. From system* (30), *we have*

$$\begin{aligned} \mathbf{f_1}(\varpi, \mathbf{x}(\varpi), \mathbf{y}(\varpi)) &= -\lambda_1 \mathbf{x}(\varpi), \\ \mathbf{f_2}(\varpi, \mathbf{x}(\varpi), \mathbf{y}(\varpi)) &= \lambda_1 \mathbf{x}(\varpi) - \lambda_2 \mathbf{y}(\varpi), \\ \mathbf{h_1}(\varpi, \mathbf{x}(\varpi)) &= \alpha_1 \exp(-\mathbf{x}(\varpi)), \\ \mathbf{h_2}(\varpi, \mathbf{y}(\varpi)) &= \alpha_2 \exp(-\mathbf{y}(\varpi)), \\ \mathbf{g_1}(\varpi, \mathbf{x}(\varpi)) &= \beta_1 \sin(\mathbf{x}(\varpi)), \\ \mathbf{g_2}(\varpi, \mathbf{y}(\varpi)) &= \beta_2 \cos(\mathbf{y}(\varpi)). \end{aligned}$$

The functions $\mathbf{f_1}, \mathbf{f_2}, \mathbf{g_1}, \mathbf{g_2}, \mathbf{h_1}$ *and* $\mathbf{g_2}$ *satisfies all the conditions for the constants* $\mathbf{a} = \alpha_1, \mathbf{b} = \alpha_2, \mathbf{a_1} = \beta_1, \mathbf{b_1} = \beta_2, \mathbf{a_2} = \mathbf{b_2} = \mathbf{a_4} = \mathbf{b_4} = \lambda_1, \mathbf{a_3} = \mathbf{a_5} = 0, \mathbf{b_3} = \mathbf{b_5} = \lambda_2$. *Let,* $\tau = 1, \alpha_1 = 0.001, \alpha_2 = 0.01, \beta_1 = 0.006, \beta_2 = 0.03$, *then* $\Lambda = \max\{\mathbf{a_1}\tau + \mathbf{a}, \mathbf{b_1}\tau + \mathbf{b}\} = 0.04 < 1$. *Further, suppose that* $\lambda_1 = 0.065, \lambda_2 = 0.014$, *we have*

$$\Delta = \max\{\mathbf{a_1}\tau + \mathbf{a} + \mathbf{a_2} + \mathbf{b_2}, \mathbf{b_1}\tau + \mathbf{b} + \mathbf{a_3} + \mathbf{b_3}\} = 0.0137 < 1.$$

All assumptions of Theorem 3.1, Theorem 3.2 and Theorem 3.3 have been satisfied for uniquness, existence *and stability of solution. We deduce RK-4 method for Equation (30) as follows:*

$$\begin{cases} \mathbf{x}^{m+1} = \mathbf{x}^m + \dfrac{1}{h}\displaystyle\sum_{l=0}^{\mathbf{q}} \left[\mathbf{g_1}\left(\varpi^{l+1}, \mathbf{x}^{l+1}\right) - \mathbf{g_1}\left(\varpi^{l}, \mathbf{x}^{l}\right)\right] \\ \qquad\qquad + \mathbf{h_1}\left(\varpi^m, \mathbf{x}^m\right) + \dfrac{h^\delta}{6\delta}[\mathbf{k}_1 + 2(\mathbf{k}_2 + \mathbf{k}_3) + \mathbf{k}_4], \\ \mathbf{y}^{m+1} = \mathbf{y}^m + \dfrac{1}{h}\displaystyle\sum_{l=0}^{\mathbf{q}} \left[\mathbf{g_2}\left(\varpi^{l+1}, \mathbf{y}^{l+1}\right) - \mathbf{g_2}\left(\varpi^{l}, \mathbf{y}^{l}\right)\right] \\ \qquad\qquad + \mathbf{h_2}\left(\varpi^m, \mathbf{y}^m\right) + \dfrac{h^\delta}{6\delta}[\mathbf{p}_1 + 2(\mathbf{p}_2 + \mathbf{p}_3) + \mathbf{p}_4], \end{cases} \tag{31}$$

such that

$$\begin{aligned} \mathbf{k}_1 &= -\lambda_1 \mathbf{x}^m, \\ \mathbf{k}_2 &= -\lambda_1[\mathbf{x}^m + \frac{h^\delta}{2\delta}\mathbf{k}_1], \\ \mathbf{k}_3 &= -\lambda_1[\mathbf{x}^m + \frac{h^\delta}{2\delta}\mathbf{k}_2], \\ \mathbf{k}_4 &= -\lambda_1[\mathbf{x}^m + \frac{h^\delta}{\delta}\mathbf{k}_3], \end{aligned}$$

and

$$\begin{aligned} \mathbf{p}_1 &= \lambda_1 \mathbf{x}^m - \lambda_2 \mathbf{y}^m, \\ \mathbf{p}_2 &= \lambda_1[\mathbf{x}^m + \frac{h^\delta}{2\delta}\mathbf{p}_1] - \lambda_2[\mathbf{y}^m + \frac{h^\delta}{2\delta}\mathbf{p}_1], \\ \mathbf{p}_3 &= \lambda_1[\mathbf{x}^m + \frac{h^\delta}{2\delta}\mathbf{p}_2] - \lambda_2[\mathbf{y}^m + \frac{h^\delta}{2\delta}\mathbf{p}_2], \\ \mathbf{p}_4 &= \lambda_1[\mathbf{x}^m + \frac{h^\delta}{\delta}\mathbf{p}_3] - \lambda_2[\mathbf{y}^m + \frac{h^\delta}{\delta}\mathbf{p}_3]. \end{aligned}$$

In Figures 8-11, we presented the approximate solution of coupled hybrid system withe non-local initial conditions using the RK-4 method using various fractional orders with step size $h = 0.1$ *and* $\lambda_1 = 0.1, \lambda_2 = 0.02, \alpha_1 = 0.1, \alpha_2 = 0.2, \beta_1 = 0.3, \beta_2 = 0.4$.

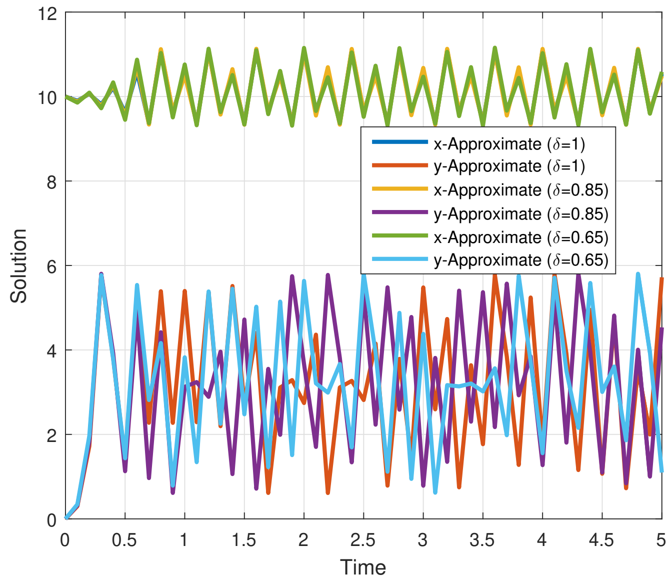

Figure 8: The approximate solution system (23).

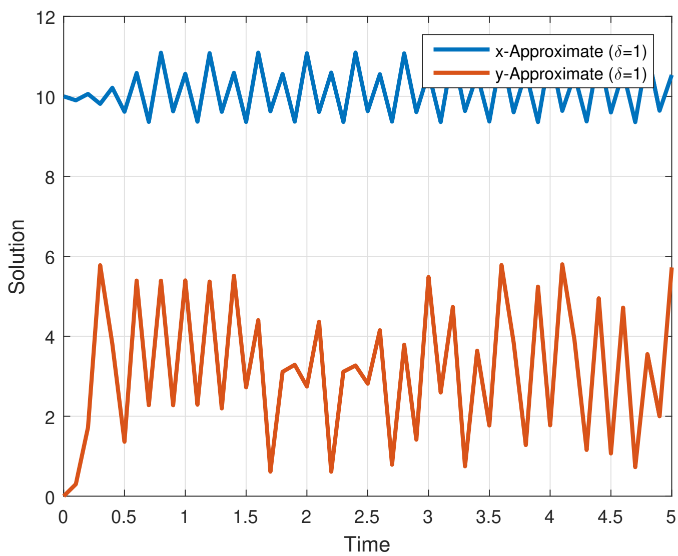

Figure 9: The approximate solution system (30).

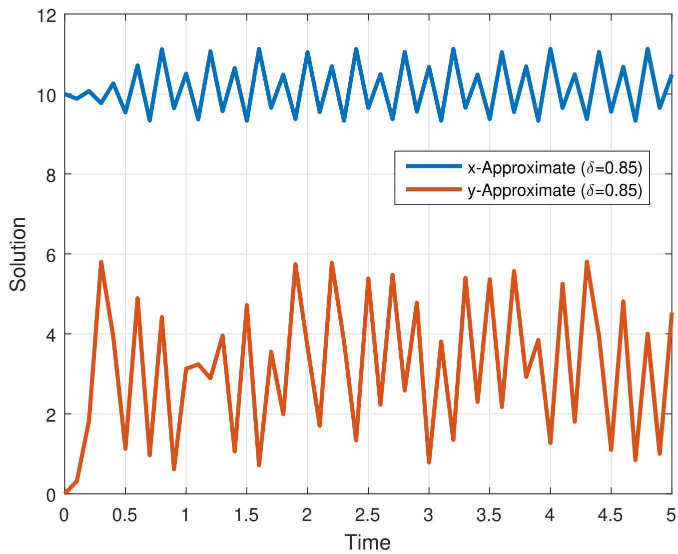

Figure 10: The approximate solution system (30).

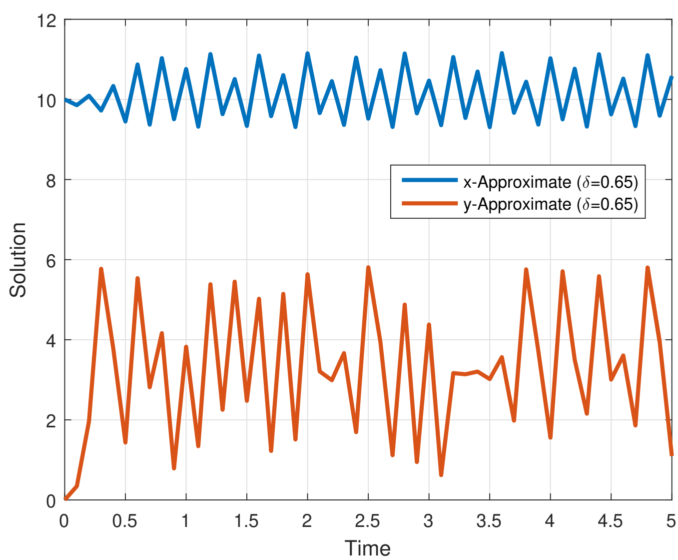

Figure 11: The approximate solution system (30).

Interestingly, if we take the initial conditions of system (30) *as* $\mathbf{x}(0) = 10^4, \mathbf{y}(0) = 0$*, then from Figure 12, the non-local system under hybrid term shows the behaviour like original problem* (23)*. Hence, non-local initial conditions effect the whole behaviour of the problem. All graphs of this study are drawn by using Matlab 2023.*

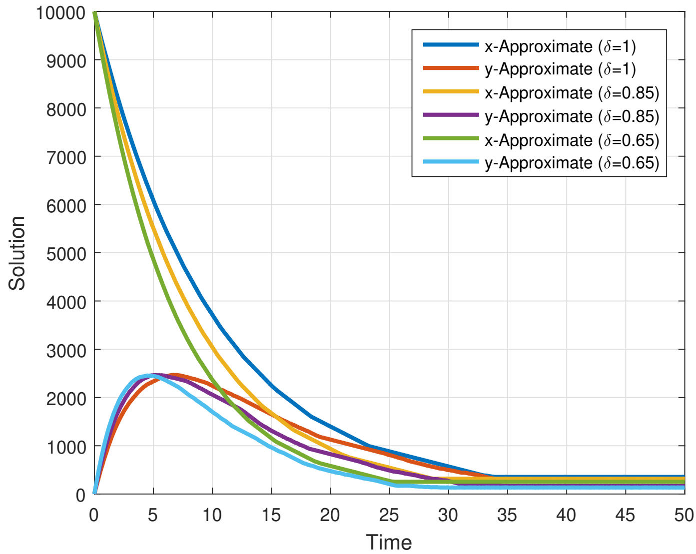

Figure 12: The approximate solution system (30).

5 Conclusion

In the current manuscript, we have investigated a hybrid type coupled system of FDEs using CFDO. First, the system was analyzed for existence theory using fixed point concept, then the U-H type stability has been discussed using the application of non-linear analysis. In addition, the fractional RK-4 scheme has been applied for the approximate results to the considered system. At the end an interesting example from real life problems was discussed in detail and justified the developed analysis. The solution of the example was also simulated graphically for different fractional order. We concluded that CFDO can also be used as a powerful differential operators for various physical problems.

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